Tensor Training

Introduction

Exercise 1

1) In a metric vector space $\tilde{a}\cdot\tilde{b}=g_{ij}a^i b^j$ with the symmetric metric **g**. With this formula show the two following properties:

i) $\tilde{a}\cdot\tilde{b}=\tilde{b}\cdot\tilde{a}$ ii) $\tilde{a}\cdot(\tilde{b}+\tilde{c})=\tilde{a}\cdot\tilde{b}+\tilde{a}\cdot\tilde{c}$.

Let's consider the completely antisymmetric unit rank 3 tensor ϵ_{ijk} with $\epsilon_{123}=1$, if you exchange two indices the sign change, for example $\epsilon_{213}=-1$, if two indices are equal the component is null, for example $\epsilon_{iki}=0$. If you keep the cycling order the sign stay the same: $\epsilon_{ijk}=\epsilon_{jki}=\epsilon_{kij}=-\epsilon_{kji}$.

2) We are in three dimensions, in an euclidean space and an rectangular system of coordinates.
We continue to use the contravariant and covariant notations as: $\vec{a}\cdot\vec{b}=a^i b_i=g_{ij}a^i b^j=a_i b^i=g^{ij}a_i b_j$.
Find the values or prove the following expressions with tensor calculus:

i) ϵ_{321} ii) ϵ^{123} iii) $\epsilon_{12k}\epsilon^{12k}$ iv) $\epsilon_{ijk}\epsilon^{ijk}$ v) $\epsilon_{imn}\epsilon^{jmn}=2\delta_i^j$ vi) $\epsilon_{ijk}\epsilon^{mnk}=\delta_i^m\delta_j^n-\delta_i^n\delta_j^m$

vii) $(\vec{a}\wedge\vec{b})_i=\epsilon_{ijk}a^j b^k$ viii) $\vec{a}\wedge\vec{a}=\vec{0}$ ix) $\vec{a}\wedge\vec{b}=-\vec{b}\wedge\vec{a}$ x) $(\vec{a}\wedge\vec{b})^i$

xi) $(\vec{u}\wedge\vec{v})\cdot\vec{w}=\vec{u}\cdot(\vec{v}\wedge\vec{w})$ xii) $\vec{a}\wedge(\vec{b}\wedge\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$

Nabla operator vector definition: $(\vec{\nabla})_i=\dfrac{\partial}{\partial x^i}=\partial_i$, $\vec{\nabla}=\vec{e}^i\partial_i$ and $\partial^i=g^{ij}\partial_j$.

Write the following expressions or prove them with tensor calculus:

i) $(\vec{\nabla}f)_i$ ii) $\vec{\nabla}f$ iii) $\vec{\nabla}.\vec{V}$ iv) $\vec{\nabla}.\vec{\nabla}f=\Delta f=\partial_i\partial^i f$ v) $(\vec{\nabla}\wedge\vec{V})^i$ vi) $(\vec{\nabla}\wedge\vec{V})_i$

vii) $(\vec{\nabla}\wedge\vec{\nabla})_i=0$ viii) $\vec{\nabla}\cdot(\vec{\nabla}\wedge\vec{V})=0$ ix) $\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{V})=\vec{\nabla}(\vec{\nabla}\cdot\vec{V})-\Delta\vec{V}$

There are a lot of operator formulas. Some are obvious, like $\vec{\nabla}(f\,g)=f\,\vec{\nabla}g+g\,\vec{\nabla}f$. Other do not turn out to be simple, as illustrated by: $\vec{\nabla}(\vec{u}\cdot\vec{v})=(\vec{u}\cdot\vec{\nabla})\vec{v}+(\vec{v}\cdot\vec{\nabla})\vec{u}+\vec{u}\wedge(\vec{\nabla}\wedge\vec{v})+\vec{v}\wedge(\vec{\nabla}\wedge\vec{u})$. Indeed $(\vec{\nabla}(\vec{u}\cdot\vec{v}))_i=\partial_i(u^j v_j)=u^j\partial_i v_j+v_j\partial_i u^j$ but, for example, $((\vec{u}\cdot\vec{\nabla})\vec{v})_i=u^j\partial_j v_i$. So to verify the formula one could develop on *x*, *y* and *z*...

Exercise 2

In Special Relativity with the rectangular metric $\eta_{\mu\nu}$, let's consider the completely antisymmetric unit rank-4 tensor $\varepsilon_{\alpha\beta\gamma\delta}$, with $\varepsilon_{0123}=1$. Find the following expressions with tensor calculus:

i) ϵ_{3210} ii) ϵ^{0123} iii) $\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\alpha\beta\gamma\delta}$ iv) $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\rho\beta\gamma\delta}$

Apply the special Lorentz transform to show that the completely antisymmetric unit rank-4 tensor is invariant in SR: $\varepsilon'^{\alpha\beta\gamma\delta}=\varepsilon^{\alpha\beta\gamma\delta}$.
Hint: the definition of the determinant can be recognized, $det\,A=\epsilon^{\alpha\beta\gamma\delta}a^0{}_\alpha a^1{}_\beta a^2{}_\gamma a^3{}_\delta$.

Exercise 3 Some demonstrations to do: i) If $T^{\mu\nu}$ is symmetric then $T_{\mu\nu}$ is also symmetric.

ii) If $T^{\mu\nu}$ symmetric then $T_{\mu}{}^{\nu} = T^{\nu}{}_{\mu}$. iii) Show that every tensors $T_{\mu\nu}$ can be considered as the sum of an antisymmetric $A_{\mu\nu}$ and symmetric $S_{\mu\nu}$ tensors ($S_{\mu\nu}=S_{\nu\mu}$ and $A_{\mu\nu}=-A_{\nu\mu}$).

iv) $S_{\mu\nu}A^{\mu\nu}=0$ v) Change of coordinates: show that $T_{\mu\nu}T^{\mu\nu}= T'_{\mu\nu}T'^{\mu\nu}$ (invariant scalar)

Exercise 4 Electromagnetism and dynamics

Electromagnetic tensor: $\boldsymbol{F}=F^{\mu\nu}=\begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$, 4-vector current: $\tilde{j}=q\,\tilde{u}$

In Special Relativity:

1) Show that the tensor $F^{\mu\nu}$ is antisymmetric.

2) Determine the components of the tensor $F_{\mu\nu}$.

3) Find the transform laws of $\vec{E}$ and $\vec{B}$ for a special Lorentz boost:

Application: In the reference frame of the laboratory *R*, we have a homokinetic beam of protons of velocity $\vec{v}$, radius *r* and density *n*. We call *R'* the proper frame of the protons at rest. With the Gauss' and Ampère's circuital laws, determine the electric and magnetic fields outside the beam in *R'*. Do the same in *R*. Can you find the same results with the Lorentz transformation shown on the right?

$$\begin{cases} E'_x=E_x \\ E'_y=\gamma(E_y-\beta c B_z) \\ E'_z=\gamma(E_z+\beta c B_y) \\ B'_x=B_x \\ B'_y=\gamma(B_y+\beta E_z/c) \\ B'_z=\gamma(B_z-\beta E_y/c) \end{cases}$$

In Special Relativity ∂_μ is a covariant 4-vector.

4) Why the following equation $\partial_\mu F^{\mu\nu}=\mu_0 j^\nu$ is covariant? An equation is said covariant, if the equation keep the same form under a transformation group, here the Lorentz group, so in all inertial frames of reference *R'*, $\partial'_\mu F'^{\mu\nu}=\mu_0 j'^\nu$. Show that this equation gives back the Maxwell equations with sources.

5) Show that the equation $\partial^\alpha F^{\mu\nu}+\partial^\mu F^{\nu\alpha}+\partial^\nu F^{\alpha\mu}=0$ is covariant, and gives back the other two Maxwell equations $\vec{\nabla}\wedge\vec{E}=-\partial\vec{B}/\partial t$ and $\vec{\nabla}\cdot\vec{B}=0$.

6) Why $F^{\mu\nu}F_{\mu\nu}$ and $\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ are Lorentz invariants? Then, show that $\vec{B}^2-\vec{E}^2/c^2$ and $\vec{E}\cdot\vec{B}$ are invariants. Use this Lorentz invariants to find again the fields in *R* from then in *R'* in the application in question 3).

7) Show that $\frac{d\vec{p}}{dt}=q(\vec{E}+\vec{v}\wedge\vec{B})$ and $\frac{dE}{dt}=q\vec{E}\cdot\vec{v}$ from the covariant equation $\frac{d\tilde{p}}{d\tau}=\boldsymbol{F}\tilde{j}$, that is, for example, $\frac{dp^\mu}{d\tau}=F^{\mu\nu}j_\nu$.